Implicit Differentiation

In Chapters 1 to 5, most functions were written in the form \( y = f(x) \), in which \( y \) was defined explicitly as a function of \( x \), such as \( y = x^3 - 4x \) and \( y = \frac{7}{x^2 + 1} \). In these equations, \( y \) is isolated on one side and is expressed explicitly as a function of \( x \).

Functions can also be defined implicitly by relations, such as the circle \( x^2 + y^2 = 25 \). In this case, the dependent variable, \( y \), is not isolated or explicitly defined in terms of the independent variable, \( x \). Since there are \( x \)-values that correspond to two \( y \)-values, \( y \) is not a function of \( x \) on the entire circle. Solving for \( y \) gives \( y = \pm \sqrt{25 - x^2} \), where \( y = \sqrt{25 - x^2} \) represents the upper semicircle and \( y = -\sqrt{25 - x^2} \) represents the lower semicircle. The given relation defines two different functions of \( x \).

Consider the problem of determining the slope of the tangent to the circle \( x^2 + y^2 = 25 \) at the point \((3, -4)\). Since this point lies on the lower semicircle, we could differentiate the function \( y = -\sqrt{25 - x^2} \) and substitute \( x = 3 \). An alternative, which avoids having to solve for \( y \) explicitly in terms of \( x \), is to use the method of implicit differentiation. Example 1 illustrates this method.

**EXAMPLE 1**

Selecting a strategy to differentiate an implicit relation

a. If \( x^2 + y^2 = 25 \), determine \( \frac{dy}{dx} \).

b. Determine the slope of the tangent to the circle \( x^2 + y^2 = 25 \) at the point \((3, -4)\).
Solution
a. Differentiate both sides of the equation with respect to \( x \).

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)
\]

\[
\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)
\]

To determine \( \frac{d}{dx}(y^2) \), use the chain rule, since \( y \) is a function of \( x \).

\[
\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy} \times \frac{dy}{dx}
\]

\[
= 2y \frac{dy}{dx}
\]

So, \( \frac{d}{dx}(x^2) + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(25) \) (Substitute)

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

b. The derivative in part a. depends on both \( x \) and \( y \). With the derivative in this form, we need to substitute values for both variables.

At the point \((3, -4)\), \( x = 3 \) and \( y = -4 \).

The slope of the tangent line to \( x^2 + y^2 = 25 \) at \((3, -4)\) is

\[
\frac{dy}{dx} = -\left( \frac{3}{-4} \right) = \frac{3}{4}.
\]
In Example 1, the derivative could be determined either by using implicit differentiation or by solving for \( y \) in terms of \( x \) and using one of the methods introduced earlier in the text. There are many situations in which solving for \( y \) in terms of \( x \) is very difficult and, in some cases, impossible. In such cases, implicit differentiation is the only algebraic method available to us.

EXAMPLE 2  Using implicit differentiation to determine the derivative

Determine \( \frac{dy}{dx} \) for \( 2xy - y^3 = 4 \).

Solution

Differentiate both sides of the equation with respect to \( x \) as follows:

\[
\frac{d}{dx}(2xy) - \frac{d}{dx}(y^3) = \frac{d}{dx}(4)
\]

Use the product rule to differentiate the first term and the chain rule to differentiate the second term.

\[
\left( \frac{d}{dx}(2x) \right) y + 2x \frac{dy}{dx} - \frac{d(y^3)}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(4)
\]

\[
2y + 2x \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0
\]

(Rearrange and factor)

\[
(2x - 3y^2) \frac{dy}{dx} = -2y
\]

(Solve for \( \frac{dy}{dx} \))

\[
\frac{dy}{dx} = -\frac{2y}{2x - 3y^2}
\]

Procedure for Implicit Differentiation

If an equation defines \( y \) implicitly as a differentiable function of \( x \), determine \( \frac{dy}{dx} \) as follows:

1: Differentiate both sides of the equation with respect to \( x \). Remember to use the chain rule when differentiating terms containing \( y \).

2: Solve for \( \frac{dy}{dx} \).

Note that implicit differentiation leads to a derivative expression that usually includes terms with both \( x \) and \( y \). The derivative is defined at a specific point on the original function if, after substituting the \( x \) and \( y \) coordinates of the point, the value of the denominator is nonzero.
**Exercise**

**PART A**

1. State the chain rule. Outline a procedure for implicit differentiation.

2. Determine \( \frac{dy}{dx} \) for each of the following in terms of \( x \) and \( y \), using implicit differentiation:
   a. \( x^2 + y^2 = 36 \)  
   b. \( 15y^2 = 2x^3 \)  
   c. \( 3xy^2 + y^3 = 8 \)  
   d. \( 9x^2 - 16y^2 = -144 \)  
   e. \( \frac{x^2}{16} + \frac{3y^2}{13} = 1 \)  
   f. \( x^2 + y^2 + 5y = 10 \)

3. For each relation, determine the equation of the tangent at the given point.
   a. \( x^2 + y^2 = 13 \), (2, -3)
   b. \( x^2 + 4y^2 = 100 \), (-8, 3)
   c. \( \frac{x^2}{25} - \frac{y^2}{36} = -1 \), (5 \( \sqrt{3} \), -12)
   d. \( \frac{x^2}{81} - \frac{5y^2}{162} = 1 \), (-11, -4)

**PART B**

4. At what point is the tangent to the curve \( x + y^2 = 1 \) parallel to the line \( x + 2y = 0 \)?

5. The equation \( 5x^2 - 6xy + 5y^2 = 16 \) represents an ellipse.
   a. Determine \( \frac{dy}{dx} \) at (1, -1).
   b. Determine two points on the ellipse at which the tangent is horizontal.

6. Determine the slope of the tangent to the ellipse \( 5x^2 + y^2 = 21 \) at the point \( A(-2, -1) \).

7. Determine the equation of the normal to the curve \( x^3 + y^3 - 3xy = 17 \) at the point (2, 3).

8. Determine the equation of the normal to \( y^2 = \frac{x^3}{2 - x} \) at the point (1, -1).

9. Determine \( \frac{dy}{dx} \):
   a. \( (x + y)^3 = 12x \)
   b. \( \sqrt{x} + y - 2x = 1 \)

10. The equation \( 4x^2y - 3y = x^3 \) implicitly defines \( y \) as a function of \( x \).
    a. Use implicit differentiation to determine \( \frac{dy}{dx} \).
    b. Write \( y \) as an explicit function of \( x \), and compute \( \frac{dy}{dx} \) directly.
    c. Show that your results for parts a. and b. are equivalent.

11. Graph each relation using graphing technology. For each graph, determine the number of tangents that exist when \( x = 1 \).
    a. \( y = \sqrt{3} - x \)
    b. \( y = -\sqrt{5} - x \)
    c. \( y = x^7 - x \)
    d. \( x^3 + 4x^2 + (x - 4)y^2 = 0 \) (This curve is known as the strophoid.)

**PART C**

12. Show that \( \frac{dy}{dx} = \frac{y}{x} \) for the relation \( \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10 \), \( x, y \neq 0 \).

13. Determine the equations of the lines that are tangent to the ellipse \( x^2 + 4y^2 = 16 \) and also pass through the point (4, 6).

14. The angle between two intersecting curves is defined as the angle between their tangents at the point of intersection. If this angle is \( 90^\circ \), the two curves are said to be orthogonal at this point.
   Prove that the curves defined by \( x^2 - y^2 = k \) and \( xy = p \) intersect orthogonally for all values of the constants \( k \) and \( p \). Illustrate your proof with a sketch.

15. Let \( l \) be any tangent to the curve \( \sqrt{x} + \sqrt{y} = \sqrt{k} \), where \( k \) is a constant. Show that the sum of the intercepts of \( l \) is \( k \).

16. Two circles of radius \( 3\sqrt{2} \) are tangent to the graph \( y^2 = 4x \) at the point (1, 2). Determine the equations of these two circles.
Related Rates

Oil that is spilled from a tanker spreads in a circle. The area of the circle increases at a constant rate of 6 km²/h. How fast is the radius of the spill increasing when the area is 9π km²? Knowing the rate of increase of the radius is important for planning the containment operation.

In this section, you will encounter some interesting problems that will help you understand the applications of derivatives and how they can be used to describe and predict the phenomena of change. In many practical applications, several quantities vary in relation to one another. The rates at which they vary are also related to one another. With calculus, we can describe and calculate such rates.

EXAMPLE 1 Solving a related rate problem involving a circular model

When a raindrop falls into a still puddle, it creates a circular ripple that spreads out from the point where the raindrop hit. The radius of the circle grows at a rate of 3 cm/s.

a. Determine the rate of increase of the circumference of the circle with respect to time.

b. Determine the rate of increase of the area of the circle when its area is 81π cm².

Solution

The radius, \( r \), and the circumference of a circle, \( C \), are related by the formula \( C = 2\pi r \).

The radius, \( r \), and the area of a circle, \( A \), are related by the formula \( A = \pi r^2 \).

We are given \( \frac{dr}{dt} = 3 \) at any time \( t \).

a. To determine \( \frac{dC}{dt} \) at any time, it is necessary to differentiate the equation \( C = 2\pi r \) with respect to \( t \), using the chain rule.

\[
\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}
\]

\[
\frac{dC}{dt} = 2\pi \frac{dr}{dt}
\]

At time \( t \), since \( \frac{dr}{dt} = 3 \),

\[
\frac{dC}{dt} = 2\pi (3) = 6\pi
\]

Therefore, the circumference is increasing at a constant rate of 6\( \pi \) cm/s.
b. To determine $\frac{dA}{dt}$, differentiate $A = \pi r^2$ with respect to $t$, using the chain rule.

\[
\frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt}
\]

\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt}
\]

We know that $\frac{dr}{dt} = 3$, so we need to determine $r$.

Since $A = 81\pi$ and $A = \pi r^2$,

\[
\pi r^2 = 81\pi
\]

\[
r^2 = 81
\]

\[
r = 9, r > 0, \text{ and } \frac{dA}{dt} = 2\pi(9)(3)
\]

\[
= 54\pi
\]

The area of the circle is increasing at a rate of $54\pi$ cm$^2$/s at the given instant.

Many related-rate problems involve right triangles and the Pythagorean theorem. In these problems, the lengths of the sides of the triangle vary with time. The lengths of the sides and the related rates can be represented quite simply on the Cartesian plane.

**EXAMPLE 2**

**Solving a related rate problem involving a right triangle model**

Natalie and Shannon start from point $A$ and drive along perpendicular roads $AB$ and $AC$, respectively, as shown. Natalie drives at a speed of 45 km/h, and Shannon travels at a speed of 40 km/h. If Shannon begins 1 h before Natalie, at what rate are their cars separating 3 h after Shannon leaves?

**Solution**

Let $x$ represent the distance that Natalie’s car has travelled along $AB$, and let $y$ represent the distance that Shannon’s car has travelled along $AC$.

Therefore, $\frac{dx}{dt} = 45$ and $\frac{dy}{dt} = 40$, where $t$ is the time in hours. (Note that both of these rates of change are positive since both distances, $x$ and $y$, are increasing with time.)

Let $r$ represent the distance between the two cars at time $t$.

Therefore, $x^2 + y^2 = r^2$.

Differentiate both sides of the equation with respect to time.

\[
\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(r^2)
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt} \text{ or } x \frac{dx}{dt} + y \frac{dy}{dt} = \frac{dr}{dt}
\]
Natalie has travelled for 2 h, or \(2 \times 45 = 90\) km. Shannon has travelled for 3 h, or \(3 \times 40 = 120\) km. The distance between the cars is
\[90^2 + 120^2 = r^2\]
\[r = 150\]

Thus, \(x = 90, \frac{dx}{dt} = 45, y = 120, \frac{dy}{dt} = 40,\) and \(r = 150\)

So,
\[
90 \times 45 + 120 \times 40 = 150 \frac{dr}{dt}
\]
\[
4050 + 4800 = 150 \frac{dr}{dt}
\]
\[
59 = \frac{dr}{dt}.
\]

Therefore, the distance between Natalie’s car and Shannon’s car is increasing at a rate of 59 km/h, 3 h after Shannon leaves.

**EXAMPLE 3**  **Solving a related rate problem involving a conical model**

Water is pouring into an inverted right circular cone at a rate of \(\pi\) m\(^3\)/min. The height and the diameter of the base of the cone are both 10 m. How fast is the water level rising when the depth of the water is 8 m?

**Solution**

Let \(V\) represent the volume, \(r\) represent the radius, and \(h\) represent the height of the water in the cone at time \(t\). The volume of the water in the cone, at any time, is \(V = \frac{1}{3}\pi r^2 h\). Since we are given \(\frac{dV}{dt}\) and we want to determine \(\frac{dh}{dt}\) when \(h = 8\), we solve for \(r\) in terms of \(h\) from the ratio determined from the similar triangles \(\frac{r}{h} = \frac{5}{10}\) or \(r = \frac{1}{2}h\). Therefore, we can simplify the volume formula so it involves only \(V\) and \(h\).

Substituting into \(V = \frac{1}{3}\pi r^2 h\), we get
\[
V = \frac{1}{3}\pi \left(\frac{1}{4}h^2\right)h
\]
\[
V = \frac{1}{12}\pi h^3
\]

Differentiating with respect to time, \(\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}\).

At a specific time, when \(h = 8\) and \(\frac{dV}{dt} = \pi\),
\[
\pi = \frac{1}{4}\pi (8)^2 \frac{dh}{dt}
\]
\[
\frac{1}{16} = \frac{dh}{dt}
\]

Therefore, at the moment when the depth of the water is 8 m, the level is rising at \(\frac{1}{16}\) m/min.

**EXAMPLE 4**  
**Solving a related rate problem involving similar triangle models**

A student who is 1.6 m tall walks directly away from a lamppost at a rate of 1.2 m/s. A light is situated 8 m above the ground on the lamppost. Show that the student’s shadow is lengthening at a rate of 0.3 m/s when she is 20 m from the base of the lamppost.

**Solution**

Let \(x\) be the length of the student's shadow, and let \(y\) be her distance from the lamppost, in metres, as shown. Let \(t\) denote the time, in seconds.

We are given that \(\frac{dy}{dt} = 1.2\) m/s, and we want to determine \(\frac{dx}{dt}\) when \(y = 20\) m.

To determine a relationship between \(x\) and \(y\), use similar triangles.

\[
\frac{x + y}{8} = \frac{x}{1.6}
\]

\[1.6x + 1.6y = 8x\]

\[1.6y = 6.4x\]

Differentiating both sides with respect to \(t\), \(1.6\frac{dy}{dt} = 6.4\frac{dx}{dt}\).

When \(y = 20\) and \(\frac{dy}{dt} = 1.2\),

\[1.6(1.2) = 6.4\frac{dx}{dt}\]

\[\frac{dx}{dt} = 0.3\]

Therefore, the student's shadow is lengthening at 0.3 m/s. (Note that her shadow is lengthening at a constant rate, independent of her distance from the lamppost.)
Exercise

PART A

1. Express the following statements in symbols:
   a. The area, \( A \), of a circle is increasing at a rate of 4 m\(^2\)/s.
   b. The surface area, \( S \), of a sphere is decreasing at a rate of 3 m\(^2\)/min.
   c. After travelling for 15 min, the speed of a car is 70 km/h.
   d. The \( x \)- and \( y \)-coordinates of a point are changing at equal rates.
   e. The head of a short-distance radar dish is revolving at three revolutions per minute.

PART B

2. The function represents the temperature, in degrees Celsius, perceived by a person standing \( x \) metres from a fire.
   a. If the person moves away from the fire at 2 m/s, how fast is the perceived temperature changing when the person is 5 m away?
   b. Using a graphing calculator, determine the distance from the fire when the perceived temperature is changing the fastest.
   c. What other calculus techniques could be used to check the result?

3. The side of a square is increasing at a rate of 5 cm/s. At what rate is the area changing when the side is 10 cm long? At what rate is the perimeter changing when the side is 10 cm long?

4. Each edge of a cube is expanding at a rate of 4 cm/s.
   a. How fast is the volume changing when each edge is 5 cm?
   b. At what rate is the surface area changing when each edge is 7 cm?

5. The width of a rectangle increases at 2 cm/s, while the length decreases at 3 cm/s. How fast is the area of the rectangle changing when the width equals 20 cm and the length equals 50 cm?

6. The area of a circle is decreasing at the rate of 5 m\(^2\)/s when its radius is 3 m.
   a. At what rate is the radius decreasing at that moment?
   b. At what rate is the diameter decreasing at that moment?

7. Oil that is spilled from a ruptured tanker spreads in a circle. The area of the circle increases at a constant rate of 6 km\(^2\)/h. How fast is the radius of the spill increasing when the area is 9\(\pi \) km\(^2\)?

8. The top of a 5 m wheeled ladder rests against a vertical wall. If the bottom of the ladder rolls away from the base of the wall at a rate of \( \frac{1}{3} \) m/s, how fast is the top of the ladder sliding down the wall when it is 3 m above the base of the wall?

9. How fast must someone let out line if a kite is 30 m high, 40 m away horizontally, and continuing to move away horizontally at a rate of 10 m/min?

10. If the rocket shown below is rising vertically at 268 m/s when it is 1220 m up, how fast is the camera-to-rocket distance changing at that instant?
11. Two cyclists depart at the same time from a starting point along routes that make an angle of $\frac{\pi}{3}$ radians with each other. The first cyclist is travelling at 15 km/h, while the second cyclist is moving at 20 km/h. How fast are the two cyclists moving apart after 2 h?

12. A spherical balloon is being filled with helium at a rate of 8 cm$^3$/s. At what rate is its radius increasing at the following moments.
   a. when the radius is 12 cm
   b. when the volume is 1435 cm$^3$ (Your answer should be correct to the nearest hundredth.)
   c. when the balloon has been filling for 33.5 s

13. A cylindrical tank, with height 15 m and diameter 2 m, is being filled with gasoline at a rate of 500 L/min. At what rate is the fluid level in the tank rising? (1 L = 1000 cm$^3$)
   About how long will it take to fill the tank?

14. If $V = \pi r^2 h$, determine $\frac{dv}{dt}$ if $r$ and $h$ are both variables that depend on $t$. In your journal, write three problems that involve the rate of change of the volume of a cylinder such that
   a. $r$ is a variable and $h$ is a constant
   b. $r$ is a constant and $h$ is a variable
   c. $r$ and $h$ are both variables

15. The trunk of a tree is approximately cylindrical in shape and has a diameter of 1 m when the height of the tree is 15 m. If the radius is increasing at 0.003 m per year and the height is increasing at 0.4 m per year, determine the rate of increase of the volume of the trunk at this moment.

16. A conical paper cup, with radius 5 cm and height 15 cm, is leaking water at a rate of 2 cm$^3$/min. At what rate is the water level decreasing when the water is 3 cm deep?

17. The cross-section of a water trough is an equilateral triangle with a horizontal top edge. If the trough is 5 m long and 25 cm deep, and water is flowing in at a rate of 0.25 m$^3$/min, how fast is the water level rising when the water is 10 cm deep at the deepest point?

18. The shadow cast by a man standing 1 m from a lamppost is 1.2 m long. If the man is 1.8 m tall and walks away from the lamppost at a speed of 120 m/min, at what rate is the shadow lengthening after 5 s?

PART C

19. A railroad bridge is 20 m above, and at right angles to, a river. A person in a train travelling at 60 km/h passes over the centre of the bridge at the same instant that a person in a motorboat travelling at 20 km/h passes under the centre of the bridge. How fast are the two people separating 10 s later?

20. Liquid is being poured into the top of a funnel at a steady rate of 200 cm$^3$/s. The funnel is in the shape of an inverted right circular cone, with a radius equal to its height. It has a small hole in the bottom, where the liquid is flowing out at a rate of 20 cm$^3$/s.
   a. How fast is the height of the liquid changing when the liquid in the funnel is 15 cm deep?
   b. At the instant when the height of the liquid is 25 cm, the funnel becomes clogged at the bottom and no more liquid flows out. How fast does the height of the liquid change just after this occurs?

21. A ladder of length $l$, standing on level ground, is leaning against a vertical wall. The base of the ladder begins to slide away from the wall. Introduce a coordinate system so that the wall lies along the positive $y$-axis, the ground is on the positive $x$-axis, and the base of the wall is the origin.
   a. What is the equation of the path followed by the midpoint of the ladder?
   b. What is the equation of the path followed by any point on the ladder? (Hint: Let $k$ be the distance from the top of the ladder to the point on the ladder.)
The Natural Logarithm and its Derivative

The logarithmic function is the inverse of the exponential function. For the particular exponential function \( y = e^x \), the inverse is \( x = e^y \) or \( y = \log_e x \), a logarithmic function where \( e \approx 2.718 \). This logarithmic function is referred to as the “natural” logarithmic function and is usually written as \( y = \ln x \).

The functions \( y = e^x \) and \( y = \ln x \) are inverses of each other. This means that the graphs of the functions are reflections of each other in the line \( y = x \), as shown.

What is the derivative of the natural logarithmic function?

For \( y = \ln x \), the definition of the derivative yields
\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}.
\]

We can determine the derivative of the natural logarithmic function using the derivative of the exponential function that we developed earlier.

Given \( y = \ln x \), we can rewrite this as \( e^y = x \). Differentiating both sides of the equation with respect to \( x \), and using implicit differentiation on the left side, yields
\[
e^y \frac{dy}{dx} = 1
\]
\[
\frac{dy}{dx} = \frac{1}{e^y}
\]
\[
= \frac{1}{x}
\]

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function \( y = \ln x \) is \( \frac{dy}{dx} = \frac{1}{x}, \, x > 0 \).

This derivative makes sense when we consider the graph of \( y = \ln x \). The function is defined only for \( x > 0 \), and the slopes are all positive. We see that, as \( x \to \infty \), \( \frac{dy}{dx} \to 0 \). As \( x \) increases, the slope of the tangent decreases.

We can apply this new derivative, along with the product, quotient, and chain rules to determine derivatives of fairly complicated functions.

**EXAMPLE 1**

*Selecting a strategy to determine the derivative of a function involving a natural logarithm*

Determine \( \frac{dy}{dx} \) for the following functions:

a. \( y = \ln(5x) \)  
b. \( y = \frac{\ln x}{x^3} \)  
c. \( y = \ln(x^2 + e^x) \)
Solution

a.  $y = \ln(5x)$

Using the chain rule,

\[ \frac{dy}{dx} = \frac{1}{5x} \cdot 5 = \frac{1}{x} \]

Using properties of logarithms,

\[ y = \ln(5x) = \ln(5) + \ln(x) \]

\[ \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x} \]

d.  $y = \frac{\ln x}{x^3}$

Using the quotient and power rules,

\[ \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x^3}\right) \]

\[ = \frac{\frac{1}{x} \cdot x^3 - \ln x \cdot 3x^2}{(x^3)^2} \]

\[ = \frac{\ln x - 3x^2 \ln x}{x^6} \]

\[ = \frac{1 - 3 \ln x}{x^4} \]

b.  $y = \frac{\ln(x^2 + e^x)}{x^3}$

Using the chain rule,

\[ \frac{dy}{dx} = \frac{1}{x^3} \cdot \frac{d}{dx}(x^2 + e^x) \]

\[ = \frac{2x + e^x}{x^3} \]

The Derivative of a Composite Natural Logarithmic Function

If $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)}g'(x)$, by the chain rule.
EXAMPLE 2  Selecting a strategy to solve a tangent problem

Determine the equation of the line that is tangent to \( y = \frac{\ln x^2}{3x} \) at the point where \( x = 1 \).

Solution

\( \ln 1 = 0 \), so \( y = 0 \) when \( x = 1 \), and the point of contact of the tangent is \((1, 0)\).

The slope of the tangent is given by \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{3x \left( \frac{1}{x^2} \right) 2x - 3 \ln x^2}{9x^2}
\]

(Quotient rule)

\[
= \frac{6 - 3 \ln x^2}{9x^2}
\]

When \( x = 1 \), \( \frac{dy}{dx} = \frac{2}{3} \).

The equation of the tangent is \( y - 0 = \frac{2}{3}(x - 1) \), or \( 2x - 3y - 2 = 0 \).

EXAMPLE 3  Determining where the minimum value of a function occurs

a. For the function \( f(x) = \sqrt{x} - \ln x \), \( x > 0 \), use your graphing calculator to determine the \( x \)-value that minimizes the value of the function.

b. Use calculus methods to determine the exact \( x \)-value where the minimum is attained.

Solution

a. The graph of \( f(x) = \sqrt{x} - \ln x \) is shown.

Use the minimum value operation of your calculator to determine the minimum value of \( f(x) \). The minimum value occurs at \( x = 4 \).

b. \( f(x) = \sqrt{x} - \ln x \)

To minimize \( f(x) \), set the derivative equal to zero.

\[
f''(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x}
\]

\[
\frac{1}{2\sqrt{x}} - \frac{1}{x} = 0
\]

\[
\frac{1}{2\sqrt{x}} = \frac{1}{x}
\]

\[
x = 2\sqrt{x}
\]

\[
x^2 = 4x
\]
\[
x(x - 4) = 0
\]
\[
x = 4 \text{ or } x = 0
\]

But \(x = 0\) is not in the domain of the function, so \(x = 4\).

Therefore, the minimum value of \(f(x)\) occurs at \(x = 4\).

We now look back at the derivative of the natural logarithmic function using the definition.

For the function \(f(x) = \ln(x)\),

\[
f'(x) = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}
\]

and, specifically,

\[
f'(1) = \lim_{h \to 0} \frac{\ln(1 + h) - \ln(1)}{h}
\]

\[
= \lim_{h \to 0} \frac{\ln(1 + h)}{h}
\]

\[
= \lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}}, \text{ since } \frac{1}{h} \ln(1 + h) = \ln(1 + h)^{\frac{1}{h}},
\]

However, we know that \(f'(x) = \frac{1}{x}, f'(1) = 1\).

We conclude that \(\lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}} = 1\).

Since the natural logarithmic function is a continuous and one-to-one function (meaning that, for each function value, there is exactly one value of the independent variable that produces this function value), we can rewrite this as \(\lim_{h \to 0} \ln[\lim_{h \to 0} (1 + h)^{\frac{1}{h}}] = 1\).

Since \(\ln e = 1\), \(\ln[\lim_{h \to 0} (1 + h)^{\frac{1}{h}}] = \ln e\).

Therefore, \(\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e\).

We now have a way to approximate the value of \(e\) using the above limit.

<table>
<thead>
<tr>
<th>(h)</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1 + h)^{\frac{1}{h}})</td>
<td>2.593 742 46</td>
<td>2.704 813 829</td>
<td>2.716 923 932</td>
<td>2.718 145 927</td>
</tr>
</tbody>
</table>

From the table, it appears that \(e \approx 2.718\) is a good approximation as \(h\) approaches zero.
Exercise

PART A

1. Distinguish between natural logarithms and common logarithms.

2. At the end of this section, we found that we could approximate the value of $e$ (Euler’s constant) using $e = \lim_{h \to 0} (1 + h)^\frac{1}{h}$. By substituting $h = \frac{1}{n}$, we can express $e$ as $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$. Justify this definition by evaluating the limit for increasing values of $n$.

3. Determine the derivative for each of the following:
   a. $y = \ln(5x + 8)$
   b. $y = \ln(x^2 + 1)$
   c. $s = 5 \ln(t^3)$
   d. $y = \ln(\sqrt{x + 1})$
   e. $s = \ln(t^3 - 2t^2 + 5)$
   f. $w = \ln(\sqrt{z^2 + 3z})$

4. Differentiate each of the following:
   a. $f(x) = x \ln x$
   b. $y = e^{\ln x}$
   c. $v = e^t \ln t$
   d. $g(z) = \ln(e^{-z} + ze^{-z})$
   e. $s = \frac{e^t}{\ln t}$
   f. $h(u) = e^{\sqrt{u}} \ln \sqrt{u}$

5. a. If $g(x) = e^{2x-1} \ln(2x - 1)$, evaluate $g'(1)$.
   b. If $f(t) = \ln\left(\frac{t - 1}{3t + 5}\right)$, evaluate $f'(5)$.
   c. Check your calculations for parts a. and b. using either a calculator or a computer.

6. For each of the following functions, solve the equation $f'(x) = 0$:
   a. $f(x) = \ln(x^2 + 1)$
   b. $f(x) = (\ln x + 2x)^\frac{1}{2}$
   c. $f(x) = (x^2 + 1)^{-1} \ln(x^2 + 1)$

7. a. Determine the equation of the tangent to the curve defined by $f(x) = \frac{\ln \sqrt{x}}{x}$ at the point where $x = 1$.
   b. Use technology to graph the function in part a. and then draw the tangent at the point where $x = 1$.
   c. Compare the equation you obtained in part a. with the equation you obtained in part b.

PART B

8. Determine the equation of the tangent to the curve defined by $y = \ln x - 1$ that is parallel to the straight line with equation $3x - 6y - 1 = 0$.

9. a. If $f(x) = (x \ln x)^2$, determine all the points at which the graph of $f(x)$ has a horizontal tangent line.
   b. Use graphing technology to check your work in part a.
   c. Comment on the efficiency of the two solutions.

10. Determine the equation of the tangent to the curve defined by $y = \ln(1 + e^{-x})$ at the point where $x = 0$.

11. The velocity, in kilometres per hour, of a car as it begins to slow down is given by the equation $v(t) = 90 - 30 \ln(3t + 1)$, where $t$ is in seconds.
   a. What is the velocity of the car as the driver begins to brake?
   b. What is the acceleration of the car?
   c. What is the acceleration at $t = 2$?
   d. How long does the car take to stop?

PART C

12. Use the definition of the derivative to evaluate $\lim_{h \to 0} \frac{\ln(2 + h) - \ln(2)}{h}$.

13. Consider $f(x) = \ln(\ln x)$.
   a. Determine $f'(x)$.
   b. State the domains of $f(x)$ and $f'(x)$.
The Derivatives of General Logarithmic Functions

In the previous section, we learned how to determine the derivative of the natural logarithmic function (base $e$). But what is the derivative of $y = \log_a x$? The base of this function is 2, not $e$.

To differentiate the general logarithmic function $y = \log_a x, a > 0, a \neq 1$, we can use the properties of logarithms so that we can use the base $e$.

Let $y = \log_a x$.

Then $a^y = x$.

Take the logarithm of both sides using the base $e$.

\[
\ln a^y = \ln x
\]
\[
y \ln a = \ln x
\]
\[
y = \frac{\ln x}{\ln a}
\]

Differentiating both sides with respect to $x$, we obtain

\[
\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)
\]
\[
= \frac{1}{\ln a} \times \frac{d}{dx}(\ln x)
\]
\[
= \frac{1}{\ln a} \times \frac{1}{x}
\]
\[
= \frac{1}{x \ln a}
\]

The Derivative of the Logarithmic Function $y = \log_a x$

If $y = \log_a x, a > 0, a \neq 1$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$. 

**EXAMPLE 1**  
**Solving a tangent problem involving a logarithmic function**

Determine the equation of the tangent to $y = \log_2 x$ at $(8, 3)$.

**Solution**

The slope of the tangent is given by the derivative $\frac{dy}{dx}$, where $y = \log_2 x$.

$$\frac{dy}{dx} = \frac{1}{x \ln 2}$$

At $x = 8$, $\frac{dy}{dx} = \frac{1}{8 \ln 2}$.

The equation of the tangent is

$$y - 3 = \frac{1}{8 \ln 2} (x - 8)$$

$$y = \frac{1}{8 \ln 2} x + 3 - \frac{1}{\ln 2}$$

We can determine the derivatives of other logarithmic functions using the rule $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, along with other derivative rules.

**EXAMPLE 2**  
**Selecting a strategy to differentiate a composite logarithmic function**

Determine the derivative of $y = \log_4(2x + 3)^5$.

**Solution**

We can rewrite the logarithmic function as follows:

$$y = \log_4(2x + 3)^5$$

$$y = 5 \log_4(2x + 3) \quad \text{(Property of logarithms)}$$

$$\frac{dy}{dx} = \frac{d}{dx}[5 \log_4(2x + 3)]$$

$$= 5 \frac{d}{dx}[\log_4(2x + 3)]$$

$$= 5 \left( \frac{1}{(2x + 3) \ln 4} \right) (2x + 3)'$$

$$= \frac{10}{(2x + 3) \ln 4}$$
The Derivative of a Composite Function Involving \( y = \log_a x \)

If \( y = \log_a f(x) \), \( a > 0 \), \( a \neq 1 \), then \( \frac{dy}{dx} = \frac{f'(x)}{f(x) \ln a} \).

**Exercise**

**PART A**

1. Determine \( \frac{dy}{dx} \) for each function.
   a. \( y = \log_5 x \)
   b. \( y = \log_3 x \)
   c. \( y = 2 \log_4 x \)
   d. \( y = -3 \log_7 x \)
   e. \( y = -(\log x) \)
   f. \( y = 3 \log_6 x \)

2. Determine the derivative of each function.
   a. \( y = \log_3 (x + 2) \)
   b. \( y = \log_8 (2x) \)
   c. \( y = -3 \log_3 (2x + 3) \)
   d. \( y = \log_{10} (5 - 2x) \)
   e. \( y = \log_8 (2x + 6) \)
   f. \( y = \log_7 (x^2 + x + 1) \)

**PART B**

3. If \( f(t) = \log_2 \left( \frac{t + 1}{2t + 7} \right) \), evaluate \( f'(3) \).
   a. If \( h(t) = \log_5 (\log_2 t) \), determine \( h'(8) \).

4. Differentiate.
   a. \( y = \log_{10} \left( \frac{1 + x}{1 - x} \right) \)
   b. \( y = \log_2 \sqrt{x^2 + 3x} \)
   c. \( y = 2 \log_3 (5^x) - \log_3 (4^x) \)
   d. \( y = 3^x \log_3 x \)
   e. \( y = 2x \log_4 x \)
   f. \( y = \frac{\log_5 (3x^2)}{\sqrt{x + 1}} \)

5. Determine the equation of the tangent to the curve \( y = x \log x \) at \( x = 10 \). Graph the function and the tangent.

6. Explain why the derivative of \( y = \log_a kx \), \( k > 0 \), is \( \frac{dy}{dx} = \frac{1}{x \ln a} \) for any constant \( k \).

7. Determine the equation of the tangent to the curve \( y = 10^{2x-9} \log_{10} (x^2 - 3x) \) at \( x = 5 \).

8. A particle’s distance, in metres, from a fixed point at time, \( t \), in seconds is given by \( s(t) = t \log_6 (t + 1) \), \( t \geq 0 \). Is the distance increasing or decreasing at \( t = 15 \)? How do you know?

**PART C**

9. a. Determine the equation of the tangent to \( y = \log_3 x \) at the point \( (9, 2) \).
   b. Graph the function and include any asymptotes.
   c. Will this tangent line intersect any asymptotes? Explain.

10. Determine the domain, critical numbers, and intervals of increase and decrease of \( f(x) = \ln(x^2 - 4) \).

11. Do the graphs of either of these functions have points of inflection? Justify your answers with supporting calculations.
   a. \( y = x \ln x \)
   b. \( y = 3 - 2 \log x \)

12. Determine whether the slope of the graph of \( y = 3^x \) at the point \( (0, 1) \) is greater than the slope of the graph of \( y = \log_3 x \) at the point \( (1, 0) \). Include graphs with your solution.
Logarithmic Differentiation

The derivatives of most functions involving exponential and logarithmic expressions can be determined by using the methods that we have developed. A function such as \( y = x^x \) poses new problems, however. The power rule cannot be used because the exponent is not a constant. The method of determining the derivative of an exponential function also cannot be used because the base is not a constant. What can be done?

It is frequently possible, with functions presenting special difficulties, to simplify the function by using the properties of logarithms. In such cases we say that we are using logarithmic differentiation.

**EXAMPLE 1** Determining the derivative of a function using logarithmic differentiation

Determine \( \frac{dy}{dx} \) for the function \( y = x^x, x > 0 \).

**Solution**

Take the natural logarithms of each side, and rewrite.

\[
\ln y = \ln x^x \]

\[
\ln y = x \ln x
\]

Differentiate both sides with respect to \( x \), using implicit differentiation on the left side and the product rule on the right side.

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \ln x
\]

\[
\frac{dy}{dx} = y(1 + \ln x)
\]

\[
= x^x(1 + \ln x)
\]

This method of logarithmic differentiation also works well to help simplify a function with many factors and powers before the differentiation takes place.

We can use logarithmic differentiation to prove the power rule, \( \frac{d}{dx}(x^n) = nx^{n-1} \), for all real values of \( n \). (In a previous chapter, we proved this rule for positive integer values of \( n \) and we have been cheating a bit in using it for other values of \( n \).)

Given the function \( y = x^n \), for any real value of \( n \) where \( x > 0 \), how do we determine \( \frac{dy}{dx} \)?
To solve this, we take the natural logarithm of both sides of the expression and get \( \ln y = \ln x^n = n \ln x \).

Differentiating both sides with respect to \( x \), using implicit differentiation, and remembering that \( n \) is a constant, we get

\[
\frac{1}{y} \frac{dy}{dx} = \frac{n}{x}
\]

\[
\frac{dy}{dx} = n y \frac{1}{x}
\]

\[
= n x^{n-1}
\]

Therefore, \( \frac{dx^n}{dx} = nx^{n-1} \) for any real value of \( n \).

**EXAMPLE 2** Selecting a logarithmic differentiation strategy to determine a derivative

For \( y = (x^2 + 3)^x \), determine \( \frac{dy}{dx} \).

**Solution**

Take the natural logarithm of both sides of the equation.

\[
y = (x^2 + 3)^x
\]

\[
\ln y = \ln (x^2 + 3)^x
\]

\[
\ln y = x \ln (x^2 + 3)
\]

Differentiate both sides of the equation with respect to \( x \), using implicit differentiation on the left side and the product and chain rules on the right side.

\[
\frac{1}{y} \frac{dy}{dx} = (1) \ln(x^2 + 3) + x \left( \frac{1}{x^2 + 3} \right) (2x)
\]

\[
\frac{dy}{dx} = y \left[ \ln(x^2 + 3) + x \left( \frac{2x}{x^2 + 3} \right) \right]
\]

\[
= (x^2 + 3)^x \left[ \ln (x^2 + 3) + \left( \frac{2x^2}{x^2 + 3} \right) \right]
\]

You will recognize logarithmic differentiation as the method used in the previous section, and its use makes memorization of many formulas unnecessary. It also allows complicated functions to be handled much more easily.
EXAMPLE 3 Using logarithmic differentiation

Given \( y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \), determine \( \frac{dy}{dx} \) at \( x = -1 \).

Solution

While it is possible to determine \( \frac{dy}{dx} \) using a combination of the product, quotient, and chain rules, this process is awkward and time-consuming. Instead, before differentiating, we take the natural logarithm of both sides of the equation.

Since \( y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \),

\[
\ln y = \ln \left[ \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \right]
\]

\[
\ln y = \ln (x^4 + 1) + \ln \sqrt{x + 2} - \ln (2x^2 + 2x + 1)
\]

\[
\ln y = \ln (x^4 + 1) + \frac{1}{2} \ln (x + 2) - \ln (2x^2 + 2x + 1)
\]

The right side of this equation looks much simpler. We can now differentiate both sides with respect to \( x \), using implicit differentiation on the left side.

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{x^4 + 1} (4x^3) + \frac{1}{2} \frac{1}{x + 2} - \frac{1}{2x^2 + 2x + 1} (4x + 2)
\]

\[
\frac{dy}{dx} = y \left[ \frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1} \right]
\]

\[
= \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \left[ \frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1} \right]
\]

While this derivative is a very complicated function, the process of determining the derivative is straightforward, using only the derivative of the natural logarithmic function and the chain rule.

We do not need to simplify this in order to determine the value of the derivative at \( x = -1 \).

For \( x = -1 \), \( \frac{dy}{dx} = \frac{(1 + 1)\sqrt{1}}{(2 - 2 + 1)} \left[ \frac{-4}{1 + 1} + \frac{1}{2(-1 + 2)} - \frac{-4 + 2}{2 - 2 + 1} \right] \)

\[
= 2 \left[ -2 + \frac{1}{2} + 2 \right]
\]

\[
= 1
\]
PART A

1. Differentiate each of the following:
   a. \( y = x^{\sqrt{10}} - 3 \)
   b. \( f(x) = 5x^3\sqrt{2} \)
   c. \( s = t^\pi \)
   d. \( f(x) = x^e + e^x \)

2. Use the method of logarithmic differentiation to determine the derivative for each of the following functions:
   a. \( y = x^{\ln x} \)
   b. \( y = \frac{(x + 1)(x - 3)^2}{(x + 2)^3} \)
   c. \( y = x^\sqrt{x} \)
   d. \( s = \left(\frac{1}{t}\right)^t \)

3. a. If \( y = f(x) = x^t \), evaluate \( f'(e) \).
    b. If \( s = e^t + t^e \), determine \( \frac{ds}{dt} \) when \( t = 2 \).
    c. If \( f(x) = \frac{(x - 3)^2\sqrt{x} + 1}{(x - 4)^3} \), determine \( f'(7) \).

4. Determine the equation of the tangent to the curve defined by \( y = x^{(x^2)} \) at the point where \( x = 2 \).

PART B

5. If \( y = \frac{1}{(x + 1)(x + 2)(x + 3)} \), determine the slope of the tangent to the curve at the point where \( x = 0 \).

6. Determine the points on the curve defined by \( y = x^\frac{3}{2}, \ x > 0 \), where the slope of the tangent is zero.

7. If tangents to the curve defined by \( y = x^2 + 4 \ln x \) are parallel to the line defined by \( y - 6x + 3 = 0 \), determine the points where the tangents touch the curve.

8. The tangent to the curve defined by \( y = x^{\sqrt{x}} \) at the point \( A(4, 16) \) is extended to cut the \( x \)-axis at \( B \) and the \( y \)-axis at \( C \). Determine the area of \( \triangle OBC \), where \( O \) is the origin.

PART C

9. Determine the slope of the line that is tangent to the curve defined by \( y = \frac{e^{x^{\sqrt{x^2}} + 1}}{(x^2 + 2)^3} \) at the point \( \left(0, \frac{1}{8}\right) \).

10. Determine \( f'(x) \) if \( f(x) = \left(\frac{x \sin x}{x^2 - 1}\right)^2 \).

11. Differentiate \( y = x^{\cos x}, \ x > 0 \).

12. Determine the equation of the line that is tangent to the curve \( y = x^x \) at the point \( (1, 1) \).

13. The position of a particle that moves on a straight line is given by \( s(t) = t^\frac{3}{2} \) for \( t > 0 \).
   a. Determine the velocity and acceleration functions.
   b. At what time, \( t \), is the velocity zero? What is the acceleration at this time?

14. Make a conjecture about which number is larger: \( e^x \) or \( x^e \). Verify your work with a calculator.
23. a. \( \left( \frac{6}{7}, \frac{2}{7}, -\frac{3}{7} \right) \)
   
   b. \( \left( \frac{-6}{7}, -\frac{2}{7}, \frac{3}{7} \right) \)

24. a. \( \overrightarrow{OC} = (8, 9) \)
   
   b. \( \overrightarrow{BD} = (10, -5) \)
   
   c. about 74.9°

25. a. \( x = t, y = -1 + t, z = 1, t \in \mathbb{R} \)
   
   b. \( (1, 2, -3) \)
   
   c. \( x = 1, y = t, z = -3 + t, t \in \mathbb{R} \)
   
   d. \( x = 1 + 3s + t, y = t, z = s, t \in \mathbb{R} \)

26. a. yes; \( x = 0, y = -1 + t, z = t, t \in \mathbb{R} \)
   
   b. no
   
   c. yes;
   
   \( x = 2 - 2t, y = t, z = 3t, t \in \mathbb{R} \)

27. 30°

28. a. \(-\frac{3}{2}\)
   
   b. 84

29. \( \vec{F} = (-1, 3, 1), t \in \mathbb{R} \)
   
   \(-x + 3y + z - 11 = 0 \)

30. \((-1, 1, 0)\)

31. a. 0.8 km
   
   b. 12 min

32. a. Answers may vary.
   
   \( \vec{F} = (6, 3, 4) + t(4, 4, 1), t \in \mathbb{R} \)
   
   b. The line found in part a will lie in the plane \( x - 2y + 4z - 16 = 0 \) if and only if both points \( A(2, -1, 3) \)
   
   and \( B(6, 3, 4) \) lie in this plane. We verify this by substituting these points into the equation of the plane, and checking for consistency. For \( A \):
   
   \( 2 - 2(-1) + 4(3) - 16 = 0 \)
   
   For \( B \):
   
   \( 6 - 2(3) + 4(4) - 16 = 0 \)
   
   Since both points lie on the plane, so does the line found in part a.

33. 20 km/h at N 53.1° E

34. parallel: 1960 N,
   
   perpendicular: about 3394.82 N

35. a. True; all non-parallel pairs of lines intersect in exactly one point in \( \mathbb{R}^2 \). However, this is not the case for lines in \( \mathbb{R}^3 \) (skew lines provide a counterexample).
   
   b. True; all non-parallel pairs of planes intersect in a line in \( \mathbb{R}^3 \).

36. a. A direction vector for \( L_1: x = 2, y - \frac{2}{3} = z = 0, 3, 1 \)
   
   and a direction vector for \( L_2: x = y + k = z + \frac{14}{k} \) is \((1, 1, k) \).
   
   But \( (0, 3, 1) \) is not a nonzero scalar multiple of \((1, 1, k) \) for any \( k \), since the first component of \((0, 3, 1) \) is \(0 \).
   
   This means that the direction vectors for \( L_1 \) and \( L_2 \) are never parallel, which means that these lines are never parallel for any \( k \).

   b. 6; \((-2, -4, -2)\)

**Calculus Appendix**

**Implicit Differentiation, p. 564**

1. The chain rule states that if \( y \) is a composite function, then \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \). To differentiate an equation implicitly, first differentiate both sides of the equation with respect to \( x \), using the chain rule for terms involving \( y \), then solve for \( \frac{dy}{dx} \).

2. a. \( \frac{x}{y} \)
   
   b. \( \frac{x^2}{5y} \)
   
   c. \( \frac{-y^2}{2xy + y^2} \)
   
   d. \( \frac{13x}{48y} \)
   
   e. \( \frac{-2x}{2y + 5} \)

3. a. \( y = \frac{2x - 13}{3} \)
   
   b. \( y = \frac{2}{3}x + 8 + 3 \)
   
   c. \( y = -\frac{3\sqrt{3x}}{5} \)
   
   d. \( y = \frac{11}{10}(x + 11) - 4 \)

4. \((0, 1)\)

5. a. 1
   
   b. \( \left( \frac{3}{\sqrt{5}}, \sqrt{5} \right) \) and \( \left( -\frac{3}{\sqrt{5}}, -\sqrt{5} \right) \)

6. \(-10\)

7. \( 7x - y - 11 = 0 \)

8. \( y = \frac{1}{2}x - \frac{3}{2} \)

9. a. \( \frac{4}{x + y} - 1 \)
   
   b. \( \sqrt{4x + y} - 1 \)

10. a. \( \frac{3x^2 - 8xy}{4x^2 - 3} \)
   
   b. \( y = \frac{x^3 - 4x^4 - 9x^2}{4x^2 - 3(4x^2 - 3)^2} \)
   
   c. \( \frac{dy}{dx} = \frac{3x^2 - 8x(4x^5 - 3)}{4x^2 - 3} \)
   
   d. \( \frac{dy}{dx} = \frac{3x^2 - 8x(4x^5 - 3)}{4x^2 - 3 - 8x^2} \)
   
   e. \( \frac{dy}{dx} = \frac{12x^4 - 9x^5 - 8x^2}{(4x^2 - 3)^2} \)
   
   f. \( \frac{dy}{dx} = \frac{4x^4 - 9x^2}{(4x^2 - 3)^2} \)

11. a. one tangent

   b. one tangent
For \( xy = P \),
\[
1 \cdot y + \frac{dy}{dx} x = P
\]
\[
\frac{dy}{dx} = \frac{-y}{x}
\]
At \( P(a, b) \),
\[
\frac{dy}{dx} = \frac{-b}{a}
\]
At point \( P(a, b) \), the slope of the tangent line of \( xy = P \) is the reciprocal of the slope of the tangent line of \( x^2 - y^2 = k \). Therefore, the tangent lines intercept at right angles, and thus, the two curves intersect orthogonally for all values of the constants \( k \) and \( P \).

15. \[
\frac{1}{2} x^4 + \frac{1}{2} y^2 \frac{dy}{dx} = 0
\]
\[
\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}
\]
Let \( P(a, b) \) be the point of tangency.

Equation on tangent line \( l \) and \( P \) is
\[
y - b = \frac{-\sqrt{b}}{\sqrt{a}}
\]
\[
x - a = \frac{-\sqrt{a}}{\sqrt{b}}
\]
x-intercept is found when \( y = 0 \).
\[
-b = \frac{-\sqrt{b}}{\sqrt{a}}
\]
\[
x = \frac{-b \sqrt{a}}{\sqrt{b}} + \frac{a \sqrt{b}}{\sqrt{a}}
\]
Therefore, the x-intercept is \( \frac{a \sqrt{b}}{\sqrt{a}} + \frac{b \sqrt{a}}{\sqrt{b}} \).

For the y-intercept, let \( x = 0 \),
\[
y - b = \frac{-\sqrt{b}}{\sqrt{a}}
\]
\[
y = \frac{-b \sqrt{a}}{\sqrt{b}} + \frac{a \sqrt{b}}{\sqrt{a}}
\]
y-intercept is \( \frac{a \sqrt{b}}{\sqrt{a}} + \frac{b \sqrt{a}}{\sqrt{b}} \).

The sum of the intercepts is
\[
a \sqrt{b} + b \sqrt{a} + a \sqrt{b} + b \sqrt{a}
\]
\[
= \frac{a b^2 + 2 a b + b^2 a}{a}
\]
\[
= \frac{a b^2}{a}
\]
\[
= a + 2 \sqrt{a} \sqrt{b} + b
\]
\[
= (a^2 + b^2)\sqrt{a}
\]
Since \( P(a, b) \) is on the curve, then
\[
\sqrt{a} + \sqrt{b} = \sqrt{k}, \text{ or } a^2 + b^2 = k^2.
\]
Therefore, the sum of the intercepts is \( (k^2)^2 = k \), as required.

16. \( (x + 2)^2 + (y - 5)^2 = 18 \) and \( (x - 4)^2 + (y + 1)^2 = 18 \)

Answers
16. $\frac{2}{\pi}$ cm/min
17. $\frac{\sqrt{3}}{4}$ m/min
18. 144 m/min
19. 62.8 km/h
20. a. $\frac{4}{5\pi}$ cm/s
   b. $\frac{8}{25\pi}$ cm/s
21. a. $x^2 + y^2 = \left(\frac{1}{2}\right)^2$
   b. $\frac{y^2}{k^2} + \frac{x^2}{(1-k)^2} = 1$

The Natural Logarithm and its Derivative, p. 575

1. A natural logarithm has base $e$; a common logarithm has base 10.
2. Since $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$, let $h = \frac{1}{n}$.
   Therefore, $e = \lim_{n \to \infty} \left(1 + \frac{1}{h}\right)^h$.
   But as $n \to 0$, $n \to \infty$.
   Therefore, $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$.
   If $n = 100$, $e = \left(1 + \frac{1}{100}\right)^{100}$
   $\approx 2.718281828...$
Try $n = 100000$, etc.

3. a. $\frac{5}{5x + 8}$
   b. $\frac{2x}{x^2 + 1}$
   c. $\frac{15}{t}$
   d. $\frac{1}{2(x + 1)}$
   e. $\frac{3r^2 - 4r}{r^3 - 2r^2 + 5}$
   f. $2(2x^2 + 3x)$

4. a. $\ln x + 1$
   b. 1
   c. $e^t \ln t + \frac{e^t}{t}$
   d. $-e^{-t}$
   e. $\frac{e^{-t} + e^{-2t}}{e^{-3t}}$
   f. $\frac{t (\ln t)^2}{2}$

5. a. 2e
   b. 0.1

The value shown is approximately 2e, which matches the calculation in part a.

This value matches the calculation in part b.

The equation on the calculator is in a different form, but is equivalent to the equation in part a.

8. $x - 2y + (2 \ln 2 - 4) = 0$
9. a. $\left(\frac{1}{x}, \frac{1}{y}\right)$ and $(1, 0)$
   b. 2
   c. The solution in part a is more precise and efficient.

The Derivatives of General Logarithmic Functions, p. 578

1. a. $\frac{1}{(x + 2) \ln 3}$
   b. $\frac{1}{x \ln 8}$
   c. $\frac{2}{(2x + 3) \ln 3}$
   d. $\frac{-2}{(5 - 2x) \ln 10}$
   e. $\frac{2x + 1}{(2x + 6) \ln 8}$
   f. $\frac{5}{(x^2 + x + 1) \ln 7}$

3. a. $\frac{1}{52 \ln 2}$
   b. $\frac{8 \log_2(8)(\ln 3)(\ln 2)}{2}$
   c. $\frac{2}{(1 - x^2) \ln 10}$
   d. $\frac{2x + 3}{2(2x^2 + 3x) \ln 2}$
   e. $\frac{2}{2 \ln 5 - \ln 4}$
   f. $\frac{\ln 3}{x \ln 3}$
   g. $\frac{x \ln 3(3^3)(\ln x) + 3^5}{x \ln 3}$

10. $y = -\frac{1}{2}x + \ln 2$
11. a. 90 km/h
   b. $-90$
   c. 3r + 1
   d. 6.36 s
12. $\frac{1}{2}$
13. a. $\frac{1}{x \ln x}$
   b. The function’s domain is $\{x \in \mathbb{R} | x > 1\}$.
   The domain of the derivative is $\{x \in \mathbb{R} | x > 0\}$ and $x \neq 1$.
Since the derivative is positive at that point, the distance is increasing at that point.

The tangent line will intersect this asymptote because it is defined for \( x = 0 \).

The point of inflection is at \( x = 0 \) and increasing for \( x < -2 \) and decreasing for \( x > 2 \).

The slope of \( y = \log_3 x \) at \( (1, 0) \) is \( \frac{1}{\ln 3} \).

Since \( \ln 3 > 1 \), the slope of \( y = 3^x \) at \( (0, 1) \) is greater than the slope of \( y = \log_3 x \) at \( (1, 0) \).

Using a calculator, \( e^\pi \approx 23.14 \) and \( \pi' \approx 22.46 \). So, \( e^\pi > \pi' \).

**Vector Appendix**

**Gaussian Elimination, pp. 588–590**

1. Answers may vary. For example:

   \[
   \begin{bmatrix}
   1 & 1.5 & 0 \\
   0 & -5.5 & 1 \\
   2 & 3 & 0 \\
   0 & -5.5 & 1 \\
   \end{bmatrix}
   \]

2. Answers may vary. For example:

   \[
   \begin{bmatrix}
   1 & 1 & 6 & 1 \\
   0 & -2 & 1 & 0 \\
   0 & 0 & -37 & 4 \\
   \end{bmatrix}
   \]

4. Answers may vary. For example:

   \[
   \begin{bmatrix}
   1 & 0 & -1 & -1 \\
   0 & -1 & 2 & 0 \\
   0 & 0 & -36 & 16 \\
   \end{bmatrix}
   \]

5. Answers may vary. For example:

   \[
   \begin{bmatrix}
   1 & 0 & -9 & -9 \\
   0 & -4 & 9 & -9 \\
   \end{bmatrix}
   \]